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Linear programming

Linear programming (LP) is a tool for solving optimization problems. It is a mathematical method for determining a way to achieve the best solution for a mathematical model of linear objective function subject to linear equality and inequality constraints where the decision variables are non-negative (positive or zero). Its feasible region is a convex polyhedron.

البرمجة الخطية هي أداة لحل مشاكل التحسين. إنها طريقة رياضية لتحديد طريقة لتحقيق أفضل حل لنموذج رياضي لدالة الهدف الخطي الخاضعة لقيود المساواة الخطية وعدم المساواة حيث تكون متغيرات القرار غير سالبة (موجبة أو صفرية). منطقتها الممكنة هي متعددة السطوح محدبة.

In 1947, George Dantzig developed an efficient method, the simplex algorithm for solving LP problems. Linear programming applications includes the following: Investment, Production planning and inventory control, Workforce planning, Urban development planning, and Oil refining and blending.

في عام 1947 ، طور جورج دانتزيغ طريقة فعالة ، وهي خوارزمية السمبلكس لحل مشاكل LP. تشمل تطبيقات البرمجة الخطية ما يلي: الاستثمار ، وتخطيط الإنتاج ومراقبة المخزون ، وتخطيط القوى العاملة ، وتخطيط التنمية الحضرية ، وتكرير النفط وخلطه.

Linear programming problem is an optimization problem for which we do the following:

1. We attempt to maximize (or minimize) a linear function of decision variables (objective function).
2. The value of decision variables must satisfy a set of constraints.
3. A sign restriction is associated with each variable. Either x_i must be nonnegative or that x_i may be unrestricted in sign (urs).

Note: LP can have unique optimal solution. Or

- Some LPs have infinite optimal solution (multiple solutions)
- Some have no feasible solutions (infeasible LP)
- Some are unbounded: there are points in the feasible region with arbitrary large (in a max problem) z-values.

Definition: A function $f(x_1, x_2, \dots, x_n)$ of the n variables $x_1, x_2, \dots, x_n$ is a linear function iff for some set of constants $c_1, c_2, \dots, c_n$,

$$f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots, c_nx_n$$

$$= c^T x$$

Definition: for any linear function $f(x_1, x_2, \dots, x_n)$ and any real number b, a linear inequality is a relation taking one or other of the following forms:

$$f(x_1, x_2, \dots, x_n) \leq b \text{ or } f(x_1, x_2, \dots, x_n) \geq b$$

Definition: for any linear function $f(x_1, x_2, \dots, x_n)$ and any real number b, the linear equality is a relation of the following

$$f(x_1, x_2, \dots, x_n) = b$$

LP general form

Maximize or Minimize :

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\leq, =, \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\leq, =, \geq) \ b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\leq, =, \geq) \ b_m \\ x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

where

(i) the function Z is the objective function.

(ii) $x_1, x_2, \dots, x_n$ are the decision variables.

(iii) the expression $(\leq, =, \geq)$ means that each constraint may take any one of the three signs.

(iv) c_j $(j = 1, \dots, n)$ represents the per unit cost or profit to the jth variable.

(v) b_i $(i = 1, \dots, m)$ is the requirement or availability of the ith constraint.

(vi) $x_1, x_2, \dots, x_n \geq 0$ is the set of non-negative restriction on the LPP. In real life problems negative decision variables have no valid meaning.

The linear programming model can be written in more efficient notation as:
Maximize or minimize

$$\mathrm{Z} = \sum_{j=1}^{n} c_j x_j$$

$$subject\ to \quad \sum_{j=1}^{n} a_{ij}\, x_j \ \leq = \geq b_i$$

$$where \quad i = 1,2,3, \dots, m, \qquad and\ j = 1,2,3, \dots, n$$

$$and \quad x_j \geq 0$$

LP problem for max and min cases can be expressed as following (**canonical form**):

$$\begin{array}{lll} max & z = c^T x \\ s.t & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{lll} min & z = c^T x \\ s.t & Ax \geq b \\ & x \geq 0 \end{array}$$

Example (The Reddy Mikks Company-two variables model)

Reddy Mikks produces both interior and exterior paints from two raw materials (مادة خام), M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, *M*1	6	4	24
Raw material, *M*2	1	2	6
Profit per ton ($1000)	5	4	

Notice that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

لاحظ أن الطلب اليومي على الطلاء الداخلي لا يمكن أن يتجاوز الطلب على الطلاء الخارجي بأكثر من 1 طن. أيضا ، الحد الأقصى للطلب اليومي على الطلاء الداخلي هو 2 طن. يريد ريدي ميكس ايجاد مزيج المنتج الأمثل (الأفضل) للدهانات الداخلية والخارجية التي تزيد من إجمالي الربح اليومي.

Solution:

The variables of the model are defined as:

x_1 = Tons produced daily of exterior paint

x_2 = Tons produced daily of interior paint

The goal is to maximize (i.e., increase as much as possible) the total daily profit of both paints. The two components of the total daily profit are expressed in terms of the variables x_1 and x_2 as:

Profit from exterior paint = $5x_1$ (thousand) dollars

Profit from interior paint = $4x_2$ (thousand) dollars

Letting z represent the total daily profit (in thousands of dollars), the objective (or goal) is expressed as

Maximize z = $5x_1 + 4x_2$

We construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed as

(Usage of a raw material by both paints) <= (Maximum raw material availability)

The daily usage of raw material M1 is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus,

Usage of raw material M1 by both paints = $6x_1 + 4x_2$ tons/day

Usage of raw material M2 by both paints = $1x_1 + 2x_2$ tons/day

The maximum daily availabilities of raw materials M1 and M2 are 24 and 6 tons, respectively. Thus, the raw material constraints are:

$6x_1 + 4x_2 <= 24$ (Raw material M1)

$x_1 + 2x_2 <= 6$ (Raw material M2)

The first restriction on product demand says that the daily production of interior paint cannot exceed that of exterior paint by more than 1 ton, which translates to:

$x_2 - x_1 <= 1$ (Market limit)

The second restriction limits the daily demand of interior paint to 2 tons that is,

$x_2 <= 2$ (Demand limit)

notice that (all) the variables, x_1 and x_2, to assume zero or positive values only. The restrictions, expressed as x_1>= 0 and x_2 >= 0, are referred to as non-negativity constraints

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\le 24 \quad (1)\\ x_1 + 2x_2 &\le 6 \quad (2)\\ -x_1 + x_2 &\le 1 \quad (3)\\ x_2 &\le 2 \quad (4)\\ x_1, x_2 &\ge 0 \quad (5) \end{aligned}$$

- Any values of x_1 and x_2 that satisfy all five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**.

For example, the solution (3, 1) is feasible because it satisfies all the constraints. On the other hand, the solution (4, 1) is infeasible because it does not satisfy at least one constraint. For example, first one.

Note: Reddy Mikks problem has an infinite number of feasible solutions, a property that is shared by all nontrivial LPs. Hence the problem cannot be solved by enumeration.

Graphical LP Solution (of two variables)

The graphical solution includes two steps:

1. Determination of the feasible solution space.

2. Determination of the optimum solution from among all the points in the solution space.

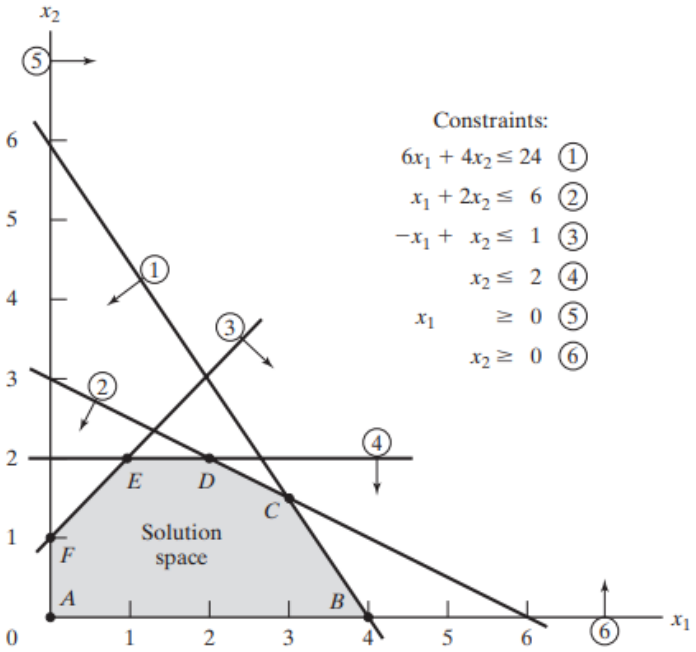
Solution of a Maximization Model

Step 1.

1. Determination of the Feasible Solution Space by graphing all the constraints including the non-negativity constraints x_1 >=0 and x_2 >= 0. The horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-paint variables, respectively.

2. Replace each inequality with an equation, and then graph the resulting straight line by locating two distinct points.

Next, consider the direction < or > of the inequality. It divides the (x_1, x_2) plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, designate any point not lying on the straight line as a reference point. If the chosen reference point satisfies the inequality, then its side is feasible; otherwise, the opposite side becomes the feasible half-space. The origin (0, 0) is a convenient reference point and should always be used so long as it does not lie on the line representing the constraint. تحديد اتجاه المتراجحة



Step 2. Determination of the Optimum Solution:

The values of x_1 and x_2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2):

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$

The solution is x_1= 3 and x_2= 1.5 with z = 5 * 3 + 4 * 1.5 = 21. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.

points	(x_1, x_2)	Values of z = $5x_1 + 4x_2$
F	(0,1)	Z=5*0+4 * 1=4
E	(1, 2)	Z=5*1+4*2=13
D	(2, 2)	Z=5*2+4*2=18
C	(3, 1.5)	Z=5*3+4*1.5=21
B	(4, 0)	Z=5*4+4*0=20

Summary of the method:

1. Graph all constraints, including non-negativity restrictions
2. Solution space consists of infinity of **feasible points**.
3. Identify **feasible corner points** of the solution space

4. Candidates for the optimum solution are given by a finite number of **corner points**
5. Use the objective function to determine the **optimum corner point** from among all the candidates

H.W: solve the following linear programming problems by using graphical method.

$$\begin{aligned} max \quad & z = 3x_1 + 2x_2 \\ subject\ to \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} max \quad & z = 5x_1 + 6x_2 \\ subject\ to \quad & x_1 + x_2 = 5 \\ & x_1 \geq 2 \\ & x_2 \leq 4 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

References:

1. Operations Research: An Introduction, Tenth Edition Hamdy A. Taha. Pearson Education, 2017.
2. Engineering Optimization: Theory and Practice, Fourth Edition Singiresu S. Rao. 2009 by John Wiley & Sons, Inc.